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A solution to the multivariable matrix factorization problem

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SUMMARY

Given a para-Hermitian matrix, $A(p_1, p_2, \ldots, p_n)$, whose elements are real, rational functions of the complex variables $p_1, p_2, \ldots, p_n$, it is shown that $A(p_1, p_2, \ldots, p_n)$ can be factorized in the form,

$$A(p_1, p_2, \ldots, p_n) = H^t(-p_1, -p_2, \ldots, -p_n)H(p_1, p_2, \ldots, p_n),$$

where the elements of the matrix $H(p_1, p_2, \ldots, p_n)$ are also real rational functions of the specified variables, if and only if $A(j\omega_1, \ldots, j\omega_n)$ is non-negative definite for all real ω_i, $i = 1, 2, \ldots, n$. A rather simple computational method for the construction of $H(p_1, p_2, \ldots, p_n)$ is given, and examples are used to illustrate how, in many cases, the factorization can actually be carried to completion with little labour.

1. Introduction

The complete solution to the single variable matrix factorization problem was given by D. C. Youla [1] in 1961, and ever since then, it has found applications in various problems of multidimensional filtering [2], control systems design [3], multiport network synthesis [15] and sampling and data reconstruction [4]. Alternate solutions to the above problem have also been presented by several other authors [5], [6], [7], and, at present, several efficient methods for obtaining the factorization in the single variable case exist. Ever since multivariable realizability theory was formally introduced in 1960 by H. Ozaki and T. Kasami [8], this has found increasing applications in synthesis of multiport lumped-distributed networks as well as in the design of microwave cavities and variable parameter networks. Koga's [9] general solution of the multiport multivariable synthesis problem requires, at one stage, factorization of a para-Hermitian non-negative definite (over the imaginary axis in the complex polydomain or hyperplane) real rational multivariable matrix (i.e. a matrix whose elements are real rational functions of several complex variables). The feasibility of factoring this type of multivariable or multiparameter matrix has not yet been demonstrated, in general. Only in some special cases (the case, for example, where the prescribed matrix is a polynomial matrix, quadratic in all the variables [9]), a solution to the factorization problem has been shown to exist but no efficient techniques for its computation have been advanced.

In the following section the multivariable matrix factorization problem is clearly stated. In section 3, the feasibility of the factorization is demonstrated by proving the existence of a solution, and a rather simple, at least in principle, iterative scheme for determining this is shown. In section 4, a certain representation for polynomials of several variables is used to actually obtain a solution, at least for certain prescribed multivariable matrices. The computational simplicity and flexibility in application of the foregoing representation is illustrated by several non-trivial examples. In the final section conclusions and recommendations for future research are made.

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2. Problem statement

The problem examined in this paper can be stated simply as follows. Let $A(p_1, p_2, \ldots, p_n)$ be a square matrix whose elements are real, rational functions of the complex variables, $p_1, p_2, \ldots, p_n$. Let $A(p_1, p_2, \ldots, p_n)$ be para-Hermitian, i.e.

$$A(p_1, p_2, \ldots, p_n) = A^t(-p_1, -p_2, \ldots, -p_n) \tag{2.1}$$

or for brevity, $A(\mathbf{p}) = A^t(-\mathbf{p})$, where

$$A(\mathbf{p}) = A(p_1, p_2, \ldots, p_n), \quad A(-\mathbf{p}) = A(-p_1, -p_2, \ldots, -p_n) \tag{2.2}$$

and the superscript 't' denotes 'transpose'. Assume that $A(j\omega_1, j\omega_2, \ldots, j\omega_n) = A(j\boldsymbol{\omega})$ is non-negative definite for all real sequences $\{\omega_i\}_{i=1}^n$. Then a matrix $H(p_1, p_2, \ldots, p_n) = H(\mathbf{p})$, composed of elements that are real rational functions of $p_1, p_2, \ldots, p_n$, is sought such that

$$A(p_1, p_2, \ldots, p_n) = H^t(-p_1, -p_2, \ldots, -p_n) H(p_1, p_2, \ldots, p_n) \tag{2.3}$$

or, again, for brevity,

$$A(\mathbf{p}) = H^t(-\mathbf{p}) H(\mathbf{p}).$$

3. Existence of a solution

The results of this section make use of an extension to a corollary of a certain theorem due to E. Artin [10]. Artin's results in this connection are first summarized below in the form of a lemma. Henceforth, it will be understood that a real rational function $a(\mathbf{p})$ will represent a scalar rational function of $p_1, p_2, \ldots, p_n$ with real coefficients.

Lemma. *Let $a(\mathbf{p})$ be a real-rational function of $\mathbf{p}$. If $a(\mathbf{p}) \geqq 0$ for all real $\mathbf{p}$ then there exist real-rational functions $f_i(\mathbf{p})$, $i = 1, 2, \ldots$, such that*

$$a(\mathbf{p}) = \sum_i (f_i(\mathbf{p}))^2, \tag{3.1}$$

for all $\mathbf{p}$.

In (3.1), the sum might not be finite in all cases. However, as pointed out by D. Hilbert [11], the sum will definitely be a finite sum at least in certain cases, and these happen to be of practical interest in engineering applications. The above lemma is extended to a theorem, which is next, stated and proved, as this theorem will be utilized later to demonstrate the existence of a solution to the multivariable matrix factorization problem.

Theorem 1. *Let $q(\mathbf{p})$ be a real-rational function of $\mathbf{p}$. If $q(j\boldsymbol{\omega}) \geqq 0$, for all real $\boldsymbol{\omega}$, where $\mathbf{p} = \boldsymbol{\sigma} + j\boldsymbol{\omega}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\omega}$ real, then there exists a real rational vector $x(\mathbf{p})]$ composed of real rational functions, such that:*

$$q(\mathbf{p}) = x(-\mathbf{p})]^t x(\mathbf{p})] \tag{3.2}$$

for all $\mathbf{p}$.

Proof: The function $q(j\boldsymbol{\omega})$ is, by hypothesis, real-rational in $\boldsymbol{\omega}$ and non-negative for all real $\boldsymbol{\omega}$. According, then, to the foregoing lemma due to E. Artin, there exist real-rational functions $f_i(\boldsymbol{\omega})$ such that

$$q(j\boldsymbol{\omega}) = \sum_i [f_i(\boldsymbol{\omega})]^2 \tag{3.3}$$

Let $f_{ie}(\boldsymbol{\omega})$ and $f_{io}(\boldsymbol{\omega})$ denote, respectively, the even and odd parts of $f_i(\boldsymbol{\omega})$. Then,

$$f_{ie}(\boldsymbol{\omega}) = \tfrac{1}{2}\{f_i(\boldsymbol{\omega}) + f_i(-\boldsymbol{\omega})\} \tag{3.4a}$$

and

$$f_{io}(\omega)=\tfrac{1}{2}\{f_i(\omega)-f_i(-\omega)\} \tag{3.4b}$$

Clearly,

$$f_{ie}(\omega)=f_{ie}(-\omega)\,, \quad \text{and} \quad f_{io}(\omega)=-f_{io}(-\omega) \tag{3.5}$$

On substituting $f_{ie}+f_{io}$ for f_i in (3.3),

$$q(j\omega)=\sum_i\left[(f_{ie})^2+(f_{io})^2+2\sum_i f_{ie}f_{io}\right], \tag{3.6}$$

which in view of (3.5) leads to

$$q(-j\omega)=\sum_i\left[(f_{ie})^2+(f_{io})^2\right]-2\sum_i f_{ie}f_{io}]. \tag{3.7}$$

it being implicitly understood that f_{ie} and f_{io} are both functions of ω. Now $q(j\omega)$ is real for all real ω and is a rational function of $j\omega$. It follows that $q(j\omega)=q(-j\omega)$ and a summation by parts of (3.6) and (3.7) yields,

$$q(j\omega)=\Sigma\left[(f_{ie})^2+(f_{io})^2\right] \tag{3.8}$$

From (3.3) and (3.5) it is evident that f_{ie} and f_{oe} are, respectively even and odd in ω. Hence, real rational functions $g_{ie}(\mathbf{p})$ and $g_{io}(\mathbf{p})$ exist such that,

$$f_{ie}(\omega)=g_{ie}(j\omega)\,,\quad f_{io}(\omega)=jg_{io}(j\omega) \tag{3.9}$$

Thus, (3.8) can be rewritten in the following form.

$$q(j\omega)=\sum_i\left[g_{ie}(j\omega)\right]^2-\sum_i\left[g_{io}(j\omega)\right]^2. \tag{3.10}$$

Moreover, as (3.5) and (3.9) clearly imply that,

$$g_{ie}(-j\omega)=g_{ie}(j\omega)\,,\quad g_{io}(-j\omega)=-g_{io}(j\omega) \tag{3.11}$$

an alternate form of (3.10) is

$$q(j\omega)=\sum_i\left[g_{ie}(j\omega)g_{ie}(-j\omega)+g_{io}(j\omega)g_{io}(-j\omega)\right] \tag{3.12}$$

Using the principle of analytic continuation in complex variable theory [12], (3.12) is readily extended to the entire complex hyperplane or polydomain.

$$q(\mathbf{p})=\sum_i\left[g_{ie}(\mathbf{p})g_{ie}(-\mathbf{p})+g_{io}(\mathbf{p})g_{io}(-\mathbf{p})\right] \tag{3.13}$$

Define, $x(\mathbf{p})]=[g_{1e}(\mathbf{p})g_{1o}(\mathbf{p})\ldots g_{ie}(\mathbf{p})g_{io}(\mathbf{p})\ldots]^t$. The vector $x(\mathbf{p})]$ is obviously comprised of real-rational functions as elements, and (3.13), when expressed in terms of $x(\mathbf{p})]$ is identical to (3.2).

It is now possible to establish the existence of a solution $H(\mathbf{p})$ to (2.1) and develop a scheme for its calculation.

Theorem 2. *Let $A(\mathbf{p})$ be a real-rational $m\times m$ multivariable matrix with normal rank r. Also it is given that $A(\mathbf{p})=A^t(-\mathbf{p})$ (i.e. $A(\mathbf{p})$ is para-Hermitian) and $A(j\omega)\geqq 0$ (i.e. $A(j\omega)$ is n.n.d) for all real ω. Assume that none of the first r nested principal minors of $A(\mathbf{p})$ is identically zero. There exists a positive integer s, $s\geqq r$ and a $s\times m$ real, rational matrix $H(\mathbf{p})$ such that*

$$H^t(-\mathbf{p})H(\mathbf{p})=A(\mathbf{p})\,, \tag{3.14}$$

for all $\mathbf{p}$, the only assumption being that the summation in the previous theorem consists of a finite number of component terms. Otherwise, the matrix H has an infinite number of rows.

Proof: The theorem will be proved by means of actual construction of a solution H of (3.14). The construction technique is an adaptation of the "Gauss diagonalization" scheme [13], [14], and starts with the working assumption that a solution of (3.14) is of the form given by (3.15), where the scalar f_{ik} and the vectors $x_i]$ are multivariable real, rational in $\mathbf{p}$.

$$H(\mathbf{p}) = \begin{bmatrix} x_1] & f_{12}x_1] & f_{13}x_1] \ldots \ldots f_{1m}x_1] \\ 0 & x_2] & f_{23}x_2] \ldots \ldots f_{2m}x_2] \\ \vdots & & \\ 0 \ldots \ldots 0 & & x_r] \ldots \ldots f_{rm}x_r] \end{bmatrix} \tag{3.15}$$

A direct substitution of (3.15) in (3.14) yields the following systems of equations.

$$a_{ik} = \sum_{j=1}^{i-1} f_{ji}(-\mathbf{p})f_{jk}(\mathbf{p})x_j(-\mathbf{p})]^t x_j(\mathbf{p})] + f_{ik}(\mathbf{p})x_i(-\mathbf{p})]^t x_i(\mathbf{p})]\,, \quad \text{for all } k > i\,, \tag{3.16}$$

$$a_{ii} = \sum_{j=1}^{i-1} f_{ji}(-\mathbf{p})f_{ji}(\mathbf{p})x_j(-\mathbf{p})]^t x_j(\mathbf{p})] + x_i(-\mathbf{p})]^t x_i(\mathbf{p})]\,, \quad \text{for all } i\,, \tag{3.17}$$

where a_{ik} denotes the element in the ith row and kth column of A. Equations represented by (3.16) are easily seen to afford a unique solution for each f_{ik} in terms of the elements $\{a_{ij}\}$ of A and the inner products $\{x_i(-\mathbf{p})]^t x_i(\mathbf{p})]\}$. An iterative method for the solution of (3.16) is possible and computationally advantageous. For $i=1$, (3.16) reduces to

$$a_{1k} = f_{1k}x_1(-\mathbf{p})]^t x_1(-\mathbf{p})]\,. \tag{3.18}$$

Whence, all f_{1k}, $k=2, 3, \ldots, m$ are immediately determined. The factors f_{2k}, $k=3, 4, \ldots, m$ are determined from

$$a_{2k} = f_{12}(-\mathbf{p})f_{1k}(\mathbf{p})x_1(-\mathbf{p})]^t x_1(\mathbf{p})] + f_{2k}x_2(-\mathbf{p})]^t x_2(\mathbf{p})] \tag{3.19}$$

and so forth, until finally f_{rk}, $k=r+1, \ldots, m$, are determined from

$$a_{rk} = \sum_{j=1}^{r-1} f_{jr}(-\mathbf{p})f_{jk}(\mathbf{p})x_j(-\mathbf{p})]^t x_j(\mathbf{p})] + f_{rk}x_r(-\mathbf{p})]^t x_r(\mathbf{p})] \tag{3.20}$$

On substituting the above values of $\{f_{ik}\}$ in (3.17), one can easily arrive at the following set of equations for $\{x_i\}$.

$$\begin{aligned} &x_1(-\mathbf{p})]^t x_1(\mathbf{p})] = d_1 \\ &x_2(-\mathbf{p})]^t x_2(\mathbf{p})] = d_2/d_1 \\ &\vdots \\ &x_r(-\mathbf{p})]^t x_r(\mathbf{p})] = d_r/d_{r-1}\,. \end{aligned} \tag{3.21}$$

where $d_1(\mathbf{p}), d_2(\mathbf{p}), \ldots, d_r(\mathbf{p})$, are the first r nested principal minors of A. In order to prove the existence of real, rational solutions $x_1], \ldots, x_r]$ to (3.21) it is noted that $A(j\omega) \geqq 0$, and $A(\mathbf{p}) = A^t(-\mathbf{p})$, implying corresponding properties for each d_i; i.e. $d_i(\mathbf{p}) = d_i(-\mathbf{p})$, $d_i(j\omega) \geqq 0$, $i=1, 2, \ldots, r$. Thus $d_1(j\omega) \geqq 0$, and $d_i(j\omega)/d_{i-1}(j\omega) \geqq 0$, $i=2, 3, \ldots, r$, for all real ω. The existence of solutions to $x_1], x_2], \ldots, x_r]$ to (3.21) is therefore guaranteed and the proof of the theorem is complete.

The assumption made in the above theorem that none of the first r nested principal minors $d_1, \ldots, d_r$ of A be identically zero is obviously necessary for (3.16) to yield finite solutions. This assumption, however, entails no real loss of generality since any $A(\mathbf{p})$ is congruent to a matrix for which the assumption is valid. Specifically, given a real, rational para-Hermitian $m \times m$ matrix $A(\mathbf{p})$ with normal rank r there exists a real, rational, $m \times m$ matrix $A_1(\mathbf{p})$ with normal rank r with none of its first r nested principal minors identically zero and such that [15]

$$A(\mathbf{p}) = Q^t A_1(\mathbf{p}) Q \tag{3.22}$$

for all $\mathbf{p}$, where Q is an $m \times m$ constant nonsingular matrix. The properties $A(\mathbf{p}) = A^t(-\mathbf{p})$, $A(j\omega) \geqq 0$, for A imply the identical properties for A_1. Thus A_1 satisfies all requirements of theorem 2, and can be expressed in the form $A_1 = H_1^t(-\mathbf{p})H_1(\mathbf{p})$, which immediately yields a factorization for A:

$$A = H^t(-\mathbf{p})H(\mathbf{p}), \tag{3.22a}$$

where

$$H(\mathbf{p}) = QH_1(\mathbf{p}). \tag{3.22b}$$

To summarize the preceding results, the general procedure for the factorization of A consists of five steps.

(1) Determine the permutation matrix Q and the matrix $A_1(\mathbf{p})$ in (3.22) so that none of the first r nested principal minors of A_1 is identically zero
(2) To factor $A_1(\mathbf{p})$, first solve (3.21) for $x_1]$, $x_2]$, ..., $x_r]$
(3) Iteratively determine the scalars $\{f_{ik}\}$ by using the recurrence relations (3.16)
(4) Form the matrix $H_1(\mathbf{p})$ as in (3.15)
(5) The solution $H(\mathbf{p})$ of (3.14) is $H(\mathbf{p}) = QH_1(\mathbf{p})$.

4. Use of quadratic form representation to facilitate computation

In the preceding section it has been noticed that the elements of $H(\mathbf{p})$ are real, rational functions in $\mathbf{p}$. In a certain special case, when the elements of $A(\mathbf{p})$ are multivariable real polynomials, quadratic in each of the variables, it was proved by T. Koga using a different technique that the factored matrix $H(\mathbf{p})$ has as elements multivariable real polynomials which are linear in the variables under consideration. Koga's results have been very nicely summarized by D. C. Youla [16]. It is, however, seen that even in this special case a considerable amount of computational effort is required to get $H(\mathbf{p})$. Here, an approach to obtain the matrix $H(\mathbf{p})$ from a wider class of multivariable, real, rational matrices than those covered by Koga, will be discussed. The basis of this approach is the modified form of a certain representation referred to as the quadratic form representation of polynomials of several variables [17]. The relevant representation is summarized below in the form of an assertion.

Assertion 1. *A real polynomial of several variables,*

$$q(\mathbf{p}) = \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{m_2} \cdots \sum_{k_n=0}^{m_n} C_{k_1k_2\ldots k_n} p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}, \tag{4.1}$$

where

$$0 \leqq \sum_{i=1}^{n} k_i \leqq \sum_{i=1}^{n} m_i$$

has associated with it a real, symmetric constant matrix B such that,

$$q(\mathbf{p}) = y(\mathbf{p})]^t By(\mathbf{p})], \tag{4.2}$$

where $y(\mathbf{p})]^t$ is a row matrix having as elements, functions, which are products of powers of $p_1, p_2, \ldots, p_n$.

The second assertion which is a modified form of the first assertion quoted from [17] is the one that is relevant in this content. As the second assertion is readily derivable from the first, no proofs are given, but illustrative examples to be included further clarify this point.

Assertion 2. *The real polynomial, $q(\mathbf{p})$ of several variables in* (4.1) *has associated with it a real symmetric constant matrix C such that,*

$$q(\mathbf{p}) = y(-\mathbf{p})]^t Cy(\mathbf{p})] \tag{4.3}$$

(4.3) *will be referred to as the modified version of the quadratic form representation.*

It is readily appreciated that in the row matrix $y]^t$ and the constant matrix C are not unique for a given $q(\mathbf{p})$. It will, further, be seen that for a fixed y, the constant symmetric matrix C can be non-unique. Sometimes, the flexibility in the construction of matrix C can be used to advantage. In (4.3) if C is non-negative definite (n.n.d), then it can be written in the standard form

$$C = D^t D \tag{4.4}$$

where D is another real constant matrix. Then substituting (4.4) in (4.3)

$$\begin{aligned} q(\mathbf{p}) &= [Dy(-\mathbf{p})]]^t [Dy(\mathbf{p})]] \\ &= \Sigma\, h_i(\mathbf{p})\, h_i(-\mathbf{p}) \,, \end{aligned} \tag{4.5}$$

where the $h_i(\mathbf{p})$'s are multivariable real polynomials. If C is not non-negative definite, the non-uniqueness of the representation for $q(\mathbf{p})$ can be sometimes used to advantage as the following example will illustrate. A single variable polynomial can be chosen to serve this purpose equally well.

Example. Consider the single variable even polynomial, $q(p_1) = -p_1^6 - p_1^4 + 4p_1^2 + 6$, where $q(j\omega_1) \geqq 0$, for all real $\omega_1\, q(p_1)$ can be written in the form,

$$\begin{aligned} q(p_1) &= y^t(-p_1)]\, B y(p_1)] \\ &= [-p_1^3, -p_1, 1] \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} p_1^3 \\ p_1 \\ 1 \end{bmatrix} . \end{aligned} \tag{4.6}$$

Certainly B is not n.n.d. in (4.6). The trouble causing term is the coefficient of p_1^2. However, by changing $y(p_1)]$, it is seen that,

$$q(p_1) = [-p_1^3, p_1^2, -p_1, 1] \begin{bmatrix} 1 & 0 & \frac{5}{4} & 0 \\ 0 & \frac{3}{2} & 0 & 3 \\ \frac{5}{4} & 0 & 2 & 0 \\ 0 & 3 & 0 & 6 \end{bmatrix} \begin{bmatrix} p_1^3 \\ p_1^2 \\ p_1 \\ 1 \end{bmatrix} . \tag{4.7}$$

In this case the dimension of the $y]$ vector has been increased to obtain an associated non-negative definite square matrix in the modified quadratic form representation of $q(p_1)$. (4.7) along with (4.4) can then be used to factor $q(p_1)$ in the desired form (4.5).

It is appreciated that it is not always necessary to increase the dimension of $y]$ in order to obtain an associated matrix, which is n.n.d. This statement is justified by the next example.

Example. Consider the two-variable quadratic form,

$$\begin{aligned} q(p_1, p_2) &= 2p_1^4 - 2p_1^3 p_2 - 0.5 p_1^2 p_2^2 + 2p_2^3 p_1 + 2p_1^4 \\ &= y(-p_1, -p_2)]^t\, B y(p_1, p_2)] \end{aligned} \tag{4.8}$$

In (4.8), if $y(-p_1, -p_2)]^t = [p_1^2, p_1 p_2, p_2^2]$, then

$$B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & -0.5 & 1 \\ 0 & 1 & 2 \end{bmatrix} . \tag{4.9}$$

Obviously this particular B is not n.n.d. By careful inspection, an alternate representation involving a different matrix B_1 can be obtained without increasing the dimension of $y]$ such that B_1 is non-negative definite (n.n.d.):

$$B_1 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1.5 & 1 \\ -1 & 1 & 2 \end{bmatrix} . \tag{4.10}$$

In (4.10), B_1 is n.n.d. and is such that

$$q(p_1, p_2) = y(-p_1, -p_2)]^t B_1 y(p_1, p_2)] \tag{4.11}$$

(4.11) can again be reduced to a form similar to (4.5), using the routine procedure to factorize n.n.d., B_1, as $B_1 = C_1^t C_1$.

The iterative scheme discussed in section 3 involves successive factorization of suitable rational functions as covered by theorem 1. In the preceding discussion, the modified form of the quadratic form representation has been used to factor, even, real, n.n.d. polynomials. However, this is no restriction as real, rational functions can also be similarly handled. For example if, $q(\mathbf{p}) = q_1(\mathbf{p})/q_2(\mathbf{p})$ is a real rational function, where $q_1(\mathbf{p})$ and $q_2(\mathbf{p})$ are real multivariable polynomials, such that $q(j\boldsymbol{\omega}) \geqq 0$, and $q(\mathbf{p}) = q(-\mathbf{p})$ then after rewriting $q(\mathbf{p})$ as

$$q(\mathbf{p}) = \frac{q_1(\mathbf{p})\, q_2(-\mathbf{p})}{q_2(\mathbf{p})\, q_2(-\mathbf{p})} = \frac{h_1(\mathbf{p})}{h_2(\mathbf{p})} \tag{4.12}$$

it is seen that the denominator $h_2(\mathbf{p})$ is already in the desired form, while the numerator polynomial, $h_1(\mathbf{p})$, is such that

$$h_1(\mathbf{p}) = h_1(-\mathbf{p}) \quad \text{and} \quad h_1(j\boldsymbol{\omega}) \geqq 0\,, \quad \text{for all real } \boldsymbol{\omega}\,. \tag{4.13}$$

The properties in (4.13) follow from the prescribed properties of $q(\mathbf{p})$, $q_1(\mathbf{p})$ and $q_2(\mathbf{p})$ in (4.12), as can be verified. Consequently the real multivariable polynomial, $h_1(\mathbf{p})$ has to be factored in the form (4.5), and attempt may be made to do this, again, using the modified quadratic form representation. The next example illustrates the implementation of the construction scheme, discussed in the proof for theorem 2, on a prescribed 2×2 para-Hermitian matrix.

Example. Consider the matrix of two variables,

$$A(p_2, p_3) = \begin{bmatrix} 1-p_2^2 & p_3-p_2 \\ -(p_3-p_2) & 1-p_3^2 \end{bmatrix}. \tag{4.14}$$

Clearly $A(p_2, p_3)$ is para-Hermitian, and $A(j\omega_2, j\omega_3) \geqq 0$, for all real values of ω_2 and ω_3. Using (3.21) and the modified version of the quadratic form representation, wherever possible, to facilitate computation, it can be shown that the factored matrix $H(p_2, p_3)$ is,

$$H(p_1, p_2) = \begin{bmatrix} (1-p_2) & \dfrac{p_3-p_2}{1+p_2} \\ 0 & \dfrac{p_2 p_3 - 1}{1+p_2} \end{bmatrix}. \tag{4.15}$$

It can be verified that,

$$A(p_1, p_2) = H^t(-p_1, -p_2)\, H(p_1 p_2)\,.$$

5. Conclusions

In this paper, one of Artin's classical results has been extended. The valid proof for this extension has been given. Then, the foregoing extension of Artin's result has been used to demonstrate the existence of a solution to the multivariable matrix factorization problem. An iterative scheme for the actual construction of the solution has been presented.

Koga [9] and Youla [16] have shown that for a real multivariable scalar polynomial, $g(\mathbf{p})$, which is at most quadratic in each of the variables, it is possible to exhibit an admissible Hermitian non-negative definite matrix B such that

$$g(\mathbf{p}) = y(-\mathbf{p})]^t B\, y(\mathbf{p})] \tag{5.1}$$

where $y(\mathbf{p})]$ is a vector consisting of as elements only linear real polynomials of the variables.

From here on, using the standard technique the n.n.d. matrix B can be factored in the form $B=D^t D$ and the factorization for $g(\mathbf{p})$ completed. No such scheme is known to be valid for multivariable polynomials $g(\mathbf{p})$ which are of degree greater than two in one or more of the variables. A modified form of the quadratic form representation of polynomials of several variables has been used to actually obtain the factorization in many of these cases, rather simply. The non-uniqueness of the quadratic form representation can be used to advantage, as the illustrative examples given substantiate. This approach can also be used in the factorization of real rational multivariable functions, and as according to the iterative scheme in section 3 the matrix factorization problem reduces to factorizing successively specified rational functions satisfying known properties, the modified version of the quadratic form representation for polynomials of several variables is naturally adaptable to the obtaining of the solution in a matrix factorization problem. The disadvantage of the fact that the modified version of the quadratic form representation cannot be used to factorize any arbitrary even non-negative definite (on the imaginary axis in the complex polydomain) real multivariable polynomial is, therefore, offset by the ease with which it can be implemented and the wide scope for its application.

Though the existence of a solution to the multivariable matrix factorization problem has been demonstrated by actually presenting a scheme for constructing the factor matrix, and a practical workable approach given for actually obtaining the factor matrix in many cases, some other problems remain to be solved. For example, in the single variable ease, uniqueness of solution has been demonstrated by imposing additional constraints on the solution. In the multivariable problem, it would be desirable to investigate the condition, if any, that would "almost guarantee" $H(p_1, p_2, \ldots, p_n)$ in (2.3) (and, may be, its inverse as well) to be analytic in the open polydomain $\operatorname{Re} p_i > 0$, for $i=1, 2, \ldots, n$. This is especially difficult, in view of the fact, that unlike in the single variable case, there are real multivariable rational functions $g(\mathbf{p})$, satisfying $g(\mathbf{p})=g(-\mathbf{p})$ and $g(j\boldsymbol{\omega}) \geqq 0$ for all real $\boldsymbol{\omega}$, which appear to be non-factorizable in the form,

$$g(\mathbf{p}) = h(-\mathbf{p})]^t h(\mathbf{p})] \tag{5.2}$$

with the constraint that the elements of $h(\mathbf{p})]$ be not only real rational but also have denominator polynomials which are Hurwitz in the multivariable sense [18]. An example which tends to substantiate this fact is

$$g(p_1, p_2) = (1-p_1 p_2)^{-2}\,. \tag{5.3}$$

The existence of $h(p)]$ in (5.2) being assured, it would also be desirable to investigate the possibility of deriving a simple algorithm that will enable one to actually obtain $h(\mathbf{p})$, given an arbitrary $g(\mathbf{p})$ satisfying

$$g(\mathbf{p}) = g(-\mathbf{p}) \quad \text{and} \quad g(j\boldsymbol{\omega}) \geqq 0 \quad \text{for all real } \boldsymbol{\omega}\,. \tag{5.4}$$

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